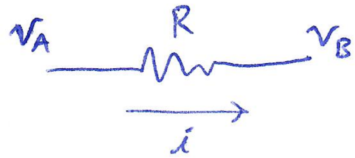
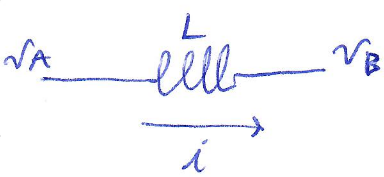
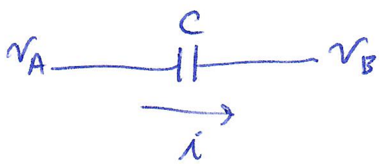
ME 4555 – Lecture 13 – Circuit analysis using TFs

Circuits are often easier to analyze in the s-domain instead of the time domain. This simply means that we look at $V(s)$, $I(s)$ instead of $v(t)$, $i(t)$.

	time domain	s-domain.
	$v_A - v_B = Ri$	$V_A - V_B = \boxed{R} I$
	$v_A - v_B = L \frac{di}{dt}$	$V_A - V_B = \boxed{Ls} I$
	$v_A - v_B = \frac{1}{C} \int i(t)\, dt$	$V_A - V_B = \boxed{\frac{1}{Cs}} I$

Each element has formula of the form $V_A - V_B = Z I$ where $Z(s)$ is called the <u>impedance</u>.

$$
\begin{cases}
Z_R = R \\
Z_L = Ls \\
Z_C = \dfrac{1}{Cs}
\end{cases}
$$

<u>Note:</u> $\mathcal{L}\left\{ \int i\, dt \right\} = \frac{1}{s} I$

Equivalently, we could start with

$$\dot{v}_A - \dot{v}_B = \frac{1}{C} i$$

$$\Big\downarrow \mathcal{L}$$

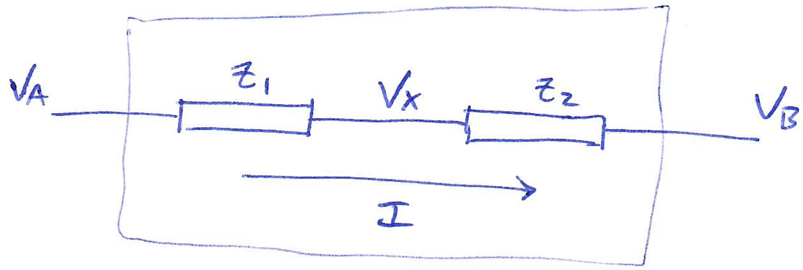
$$s(V_A - V_B) = \frac{1}{C} I$$

$$\Big\downarrow$$

$$V_A - V_B = \frac{1}{Cs} I$$

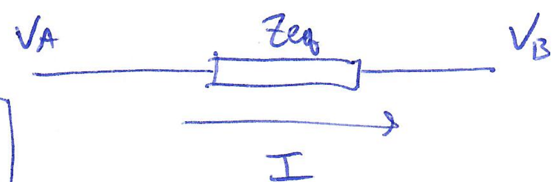
Equivalent impedances

Series connection:



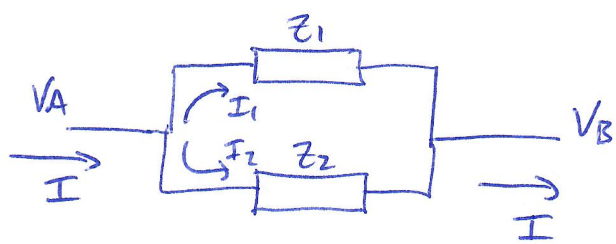
$$\begin{cases} V_A - V_x = Z_1 I \\ V_x - V_B = Z_2 I \end{cases} \Rightarrow V_A - V_B = \underbrace{(Z_1 + Z_2)}_{Z_{eq}.} I$$

so this is equivalent to



with $\boxed{Z_{eq} = Z_1 + Z_2}$

Parallel connection:

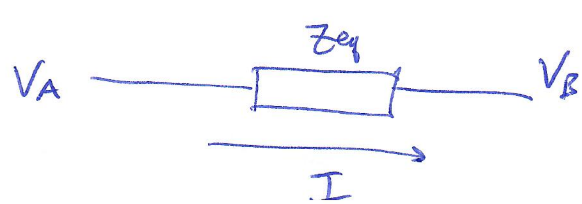


$$\begin{cases} V_A - V_B = Z_1 I_1 \\ V_A - V_B = Z_2 I_2 \\ I_1 + I_2 = I \end{cases} \Longrightarrow \underbrace{\frac{V_A - V_B}{Z_1}}_{I_1} + \underbrace{\frac{V_A - V_B}{Z_2}}_{I_2} = I$$

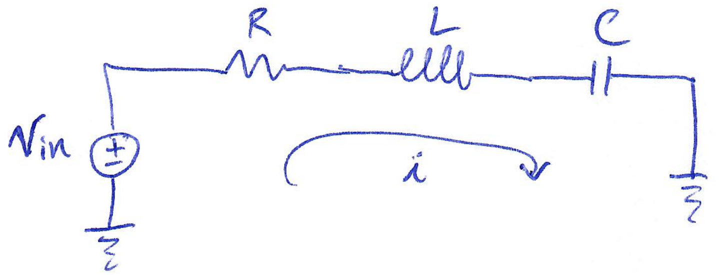
$$\Rightarrow \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)(V_A - V_B) = I$$

$$\Rightarrow V_A - V_B = Z_{eq} I$$

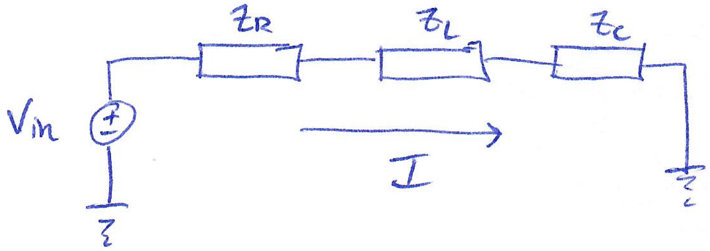
where $\boxed{\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2}}$



Ex RLC circuit.

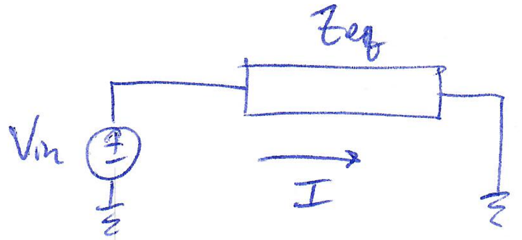


$\mathcal{L}$



where $\begin{cases} Z_R = R \\ Z_L = Ls \\ Z_C = \dfrac{1}{Cs} \end{cases}$

equivalent



where $Z_{eq} = Z_R + Z_L + Z_C$.

transfer function from $V_{in} \to I$

$$G(s) = \frac{1}{Z_{eq}} = \frac{I}{V_{in}} = \boxed{\frac{Cs}{LCs^2 + RCs + 1}}$$

so $\boxed{V_{in} = Z_{eq} I}$

$$\Rightarrow \quad V_{in} = \left(R + Ls + \frac{1}{Cs}\right) I$$

multiply by s (i.e. differentiate)

$$\Rightarrow \quad sV_{in} = \left(Ls^2 + Rs + \frac{1}{C}\right) I$$

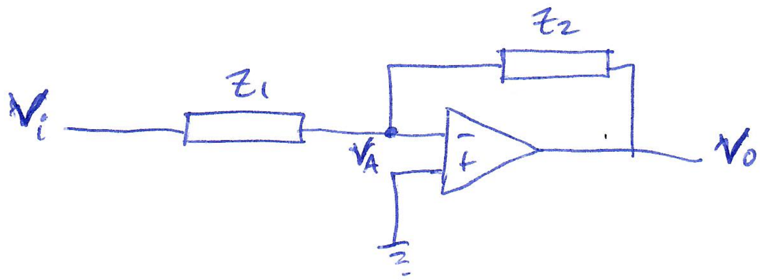
$\mathcal{L}^{-1}$ (inverse Laplace)

$$\Rightarrow \quad \boxed{\dot{V}_{in} = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C}}$$

or, if we let $Q = \frac{1}{s} I$ $\left(q(t) = \int i(t)\,dt\right.$, charge is integral of current$\left.\right)$

then $V_{in} = \left(Ls^2 + Rs + \frac{1}{C}\right) Q \Rightarrow \boxed{V_{in} = L\ddot{q} + R\dot{q} + \frac{1}{C}q}$

Ex inverting amplifier (with impedances)

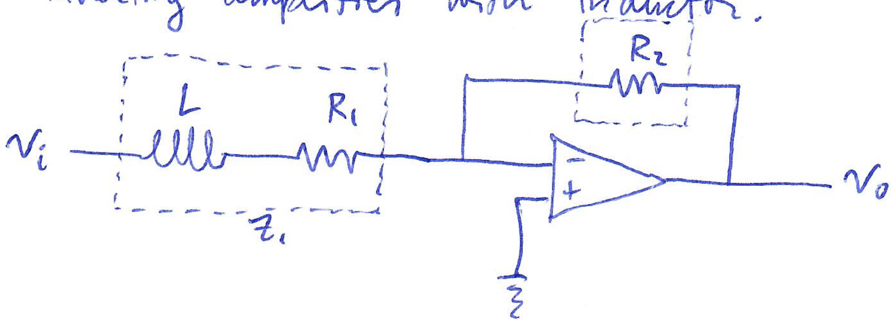


Recall:
$$V_i - V_A = Z_1 I$$
$$V_A - V_o = Z_2 I$$
$$V_A = 0$$

because no current enters the opamp in the + or − terminals and voltages are equal, so since $V_+ = 0$ (ground), $V_- = V_A = 0$ also.

Solving, we find that $\boxed{\dfrac{V_o}{V_i} = -\dfrac{Z_2}{Z_1}}$

Ex inverting amplifier with inductor.



Z_1 is series connection of inductor (Ls) and resistor (R_1).

so $Z_1 = Ls + R_1$, $Z_2 = R_2$

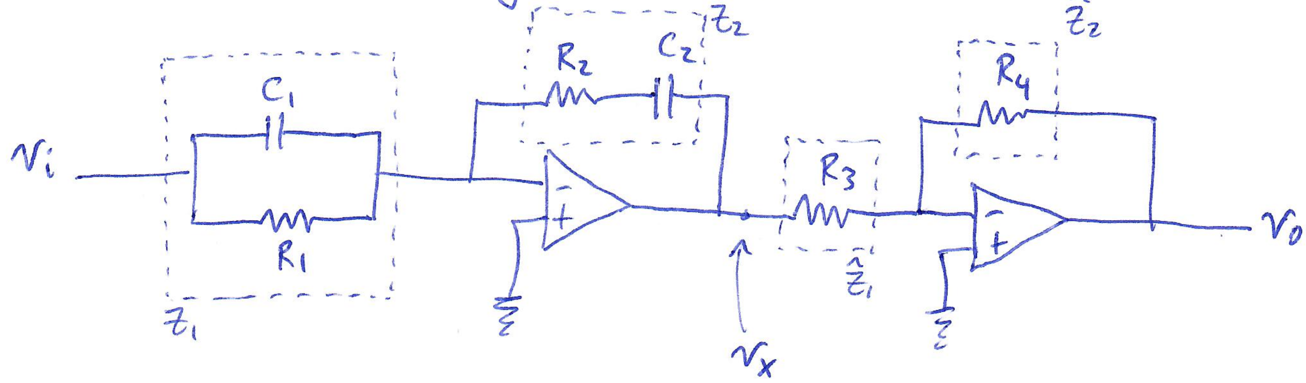
$$\Rightarrow \boxed{\frac{V_o}{V_i} = -\frac{R_2}{Ls + R_1}}$$

can also obtain ODE:

$$\frac{(Ls + R_1)}{R_2} V_o = -V_i$$

$$\Rightarrow \boxed{\frac{L}{R_2}\dot{v}_o + \frac{R_1}{R_2} v_o = -v_i}$$

Ex cascaded inverting amplifiers.



First part (from v_i to v_x):

Z_1 is $C_1 \,||\, R_1$: $\dfrac{1}{Z_1} = \dfrac{1}{\left(\frac{1}{C_1 s}\right)} + \dfrac{1}{R_1} \Rightarrow \boxed{Z_1 = \dfrac{R_1}{R_1 C_1 s + 1}}$

Z_2 is $R_2 + C_2$: $Z_2 = \dfrac{1}{C_2 s} + R_2 \Rightarrow \boxed{Z_2 = \dfrac{R_2 C_2 s + 1}{C_2 s}}$

therefore: $$\frac{v_x}{v_i} = -\frac{Z_2}{Z_1} = -\frac{(R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_1 C_2 s}$$

Second part (from v_x to v_o):

$\hat{Z}_1$ is R_3

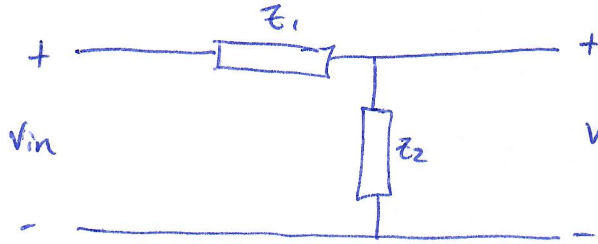
$\hat{Z}_2$ is R_4

therefore $$\frac{v_o}{v_x} = -\frac{\hat{Z}_2}{\hat{Z}_1} = -\frac{R_4}{R_3}$$

So $$\boxed{\frac{v_o}{v_i} = \left(\frac{v_o}{v_x}\right)\left(\frac{v_x}{v_i}\right) = \frac{R_4 (R_1 C_1 s + 1)(R_2 C_2 s + 1)}{R_3 R_1 C_2 s}}$$

Note: this is one way to implement a PID controller using an analog circuit! (more on this later...)

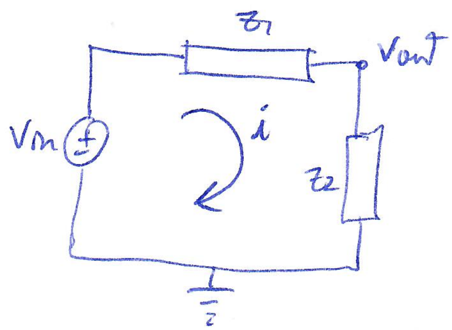
Ex



What is the transfer function $\dfrac{V_{out}}{V_{in}}$?

Does this even make sense? there is no "source"! Is there even current?

Lets put a source at V_{in}:

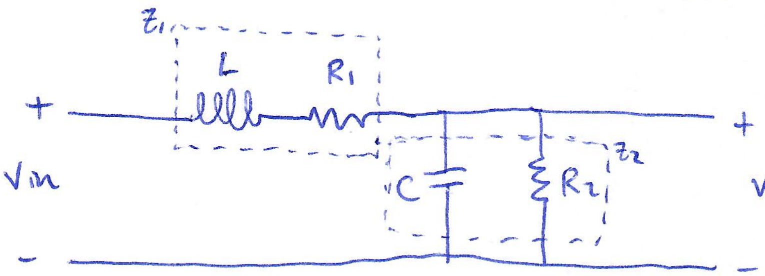


then:
$$\begin{cases} V_{in} - V_{out} = Z_1 I \\ V_{out} = Z_2 I \end{cases}$$

Substitute to eliminate I: $\quad V_{in} - V_{out} = \dfrac{Z_1}{Z_2} V_{out}$

$$\Rightarrow \boxed{\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2}}$$

Ex: model for an electric transmission line (used in mid-range power transmission)



Z_1: $\quad L + R_1 \longrightarrow Z_1 = (Ls + R_1)$

Z_2: $\quad C \,/\!/\, R_2 \longrightarrow \dfrac{1}{Z_2} = Cs + \dfrac{1}{R_2} \quad \Rightarrow \quad Z_2 = \dfrac{R_2}{R_2 Cs + 1}$

Using same result from previous example:

$$\frac{V_{out}}{V_{in}} = \frac{Z_2}{Z_1 + Z_2} = \frac{\frac{R_2}{R_2Cs+1}}{Ls + R_1 + \frac{R_2}{R_2Cs+1}} = \boxed{\frac{R_2}{(Ls+R_1)(R_2Cs+1)+R_2}}$$